Math 210A Lecture 9 Notes

Daniel Raban

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1 Equalizers, Kernels, and Ideals

1.1 Equalizers and coequalizers

Definition 1.1. Let $f, g : A \to B$ be morphisms in $\mathcal{C}$. The **equalizer** is the limit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

It satisfies the following diagram:

$$\begin{tikzcd} \mathrm{eq}(f,g) \arrow[r, "\iota"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ Y \arrow[u, dashed] \arrow[ur, "q"'] \end{tikzcd}$$

A **coequalizer** is the colimit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

It satisfies the following diagram:

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "\pi"] \arrow[dr, "q"] & \mathrm{coeq}(f,g) \arrow[d, dashed] \\ & & Y \end{tikzcd}$$

Lemma 1.1. *$\iota : \mathrm{eq}(f,g) \to A$ is a monomorphism, and $\pi : B \to \mathrm{coeq}(f,g)$ is an epimorphism.*

Proof. Let $\alpha, \beta : C \to \mathrm{eq}(f,g)$ be such that $\iota \circ \alpha = \iota \circ \beta$. Then there is a unique morphism $\phi : C \to \mathrm{eq}(f,g)$ making the following diagram commute:

$$\begin{tikzcd}
\mathrm{eq}(f,g) \arrow[r, "\iota"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
C \arrow[u, dashed, "\phi"] \arrow[ur, shift left, "\iota\circ\alpha"] \arrow[ur, shift right, "\iota\circ\beta"'] & &
\end{tikzcd}$$

But α and β satisfy the property of ϕ, so $\alpha = \phi = \beta$. The property for coequalizers follows from reversing the arrows. $\square$

Theorem 1.1. *Every category with products and equalizers is complete.*

Proof. Let $F : I \to \mathcal{C}$ be a functor. Then

$$\begin{tikzcd}
\prod_{i\in I} F(i) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \prod_{\phi: i\mapsto\phi(i)} F(\phi(i))
\end{tikzcd}$$

where f is

$$\prod_{k\in I} F(k) \xrightarrow{\pi_i} F(i) \xrightarrow{F(\phi)} F(\phi(i))$$

and g is

$$\prod_{k\in I} F(k) \xrightarrow{\pi_{\pi(i)}} F(\phi(i))$$

We claim that $\mathrm{eq}(f,g) \to \prod_{i\in I} F(i) \to F(i)$ is the limit. The

$$\begin{tikzcd}
\mathrm{eq}(f,g) \arrow[r] \arrow[dr] & F(i) \arrow[d, "F(\phi)"] \\
& F(\phi(i))
\end{tikzcd}$$

commute for all ϕ. So the equalizer has the property of the limit. To show the universal property, suppose we have the following diagram for some X.

$$\begin{tikzcd}
X \arrow[r, "\psi_i"] \arrow[dr, "\psi_{\phi(i)}"] & F(i) \arrow[d, "F(\phi)"] \\
& F(\phi(i))
\end{tikzcd}$$

This is the same as

$$\begin{tikzcd}
X \arrow[r] \arrow[d, dashed] & \prod_{i\in I} F(i) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \prod_{\phi: i\mapsto\phi(i)} F(\phi(i)) \\
\mathrm{eq}(f,g) \arrow[ur] & &
\end{tikzcd}$$

by the universal property of the equalizer. So $\mathrm{eq}(f, g)$ satisfies the universal property of $\lim F$. $\square$

Example 1.1. In Set, Gp, Ring, Rmod, and Top, the equalizer of $f, g : A \to B$ is $\mathrm{eq}(f, g) = \{x \in A : f(x) = g(x)\}$. These are all complete categories. The are also complete, as they have coproducts and coequalizers.

1.2 Kernels and ideals

Definition 1.2. A **zero object** is an object which is both initial and terminal.

Let $\mathcal{C}$ have a zero object 0. There exists a unique morphism $0 : A \to B$ which is the composition of the unique morphism from $A \to 0$ and $0 \to B$.

Definition 1.3. For $f : A \to B$, the **kernel** $\ker(f) = \mathrm{eq}(f, 0)$ and $\mathrm{coker}(f) = \mathrm{coeq}(f, 0)$, where 0 is the unique zero morphism.

Example 1.2. In Gp, $\ker(f : G \to G') = \{g \in G : f(g) = e\}$. This is the same in Rmod.

Example 1.3. In Ring, we can makes sense of this is we work in a larger category, Rng, of pseudorings (rings without identity). If $f : R \to S$, then $\ker(f) = \{x \in R : f(x) = 0\}$. In fact, $\ker f$ is a two-sided ideal.

In all of these cases, $\ker f = 0$ iff f is a monomorphism iff f is 1 to 1. To show that $\ker(f) = 0$ implies that f is a monomorphism, we have (in Gp)

$$f(g) = f(h) \implies f(gh^{-1}) = e \implies gh^{-1} = e \implies g = h,$$

but this requires internal knowledge of the structure of the category.

Proposition 1.1. *1. If $f : G \to G'$ is a homomorphism,* $\ker(f) \trianglelefteq G$.

2. If $N \trianglelefteq G$, then $N = \ker(\pi)$, where $f : G \to G/N$ sends $g \mapsto gN$.

Proof. To prove the first part, note that $f(gng^{-1}) = f(g)f(n)f(g)^{-1} = e$, so $gng^{-1} \in \ker(f)$. The second follows from the definitions. $\square$

Theorem 1.2. *Let $f : G \to G'$ be a homomorphism. Then $\overline{f} : G/\ker(f) \to \mathrm{im}(f)$ given by $\overline{f}(g\ker(f)) = f(g)$ is an isomorphism.*

Definition 1.4. A **left ideal** I of a ring R is a subgroup such that $ri \in I$ for all $r \in R$ and $i \in I$. A **right ideal** I of a ring R is a subgroup such that $is \in I$ for all $s \in R$ and $i \in I$. A **(two-sided) ideal** I is a right and left ideal.

If we have a left ideal I, left multiplication $R \times I \to R$ makes I a left R-module. So a left ideal of R is exactly a left R-submodule of R.

Definition 1.5. An (R, S)-**bimodule** M is a left R-module that is also a right S-module such that $(rm)s = r(ms)$ for all $r \in R$, $s \in S$, and $m \in M$.

A (two-sided) ideal is an (R, R)-subbimodule of R.

If $I \subseteq R$ is a two-sided idea, then $R/I = \{a + I : a \in R\}$. We have addition $(a + I) + (b + I) = (a + b) + I$ and multiplication $(a + I)(b + I) = ab + I$. Why is multiplication well-defined? For $a, b \in R$ and $i, j \in I$,

$$(a+i)(b+j) = ab + \underbrace{aj}_{\in I} + \underbrace{ib}_{\in I} + \underbrace{ij}_{\in I} \in ab + I.$$

Definition 1.6. R/I is called a **quotient ring**.

Observe that $\ker(f)$ with $f : R \to S$ is an ideal. If $a \in \ker(f)$, $r, s \in R$, then $f(ras) = f(r)f(a)f(s) = 0$. So we have the $\pi : R \to R/I$ with $\pi(r) = r + I$ and $\ker(\pi) = I$. So $R/\ker(f) \cong \operatorname{im}(f)$.

This also works with with left, right, and bimodules. In fact, it works even better! All left R-modules are kernels, so you don't need any conditions like normality.

What about cokernels? In Gp, we have a problem: if $f : G \to G'$, $\operatorname{im}(f)$ may not be normal in G'. We take $\operatorname{coker}(f) = G/\overline{\operatorname{im}(f)}$, where $\overline{\operatorname{im}(f)}$ denotes the **normal closure** of $\operatorname{im}(f)$, the smallest normal subgroup containing $\operatorname{im}(f)$.

We have been using the term image in the sense of groups. Here is a categorical point of view.

Definition 1.7. The **image** $\operatorname{im}(f)$ of $f : A \to B$ is an object and a monomorphism $\iota : \operatorname{im}(f) \to B$ such that there exists $\pi : A \to \operatorname{im}(f)$ with $\pi \circ \iota$ and such that if $e : C \to B$ is a monomorphism and $g : A \to C$ is such that $e \circ g = f$, then there exists a unique morphism $\psi : \operatorname{im}(f) \to C$ such that $g \circ \psi = \iota$.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "\pi"] \arrow[ddr, "g"'] & & B \\
& \operatorname{im}(f) \arrow[ur, "\iota"] \arrow[d, dashed, "\psi"'] & \\
& C \arrow[uur, "e"'] &
\end{tikzcd}$$

Note that $e \circ \psi \circ \pi = e \circ g \implies \psi \circ \pi = g$, since e is a monomorphism.